

Rings With Topologies Induced by Spaces of Functions

Răzvan Gelca

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Abstract: By considering topologies on Noetherian rings that carry the properties of those induced by spaces of functions, we prove that if an ideal is closed then every prime ideal associated to it is closed (thus answering a question raised in [5]). The converse is also true if we assume that a topological version of the Nullstellensatz holds, and we prove such a result for the ring of polynomials in two variables endowed with the topology induced by the Hardy space. The topological completion of the ring is a module, and we show the existence of a one-to-one correspondence between closed ideals of finite codimension and closed submodules of finite codimension which preserves primary decompositions. At the end we consider the case of the Hardy space on the polydisc, and of Bergman spaces on the unit ball and Reinhardt domains.

In the development of multivariable operator theory there appears to exist a fundamental connection between the study of invariant subspaces for certain Hilbert spaces of holomorphic functions and the study of relatively closed ideals in rings of functions that are dense in these spaces. Some results in this direction have been obtained in [5], [6] and [7]. Most of this has been done in the context of Hilbert modules. In this paper we consider rings with topologies that carry some of the properties of those mentioned

above and exhibit topological properties for ideals in these rings. This general approach will give a more transparent view of some results proved in [5].

1. Hilbert Nullstellensatz for closed ideals

Throughout the paper \mathcal{R} will denote a commutative Noetherian ring with unit, endowed with a topology τ for which addition is continuous and multiplication is separately continuous in each variable.

An example of such a ring is the ring $\mathbf{C}[z_1, z_2, \dots, z_n]$ of polynomials in n variables with the topology induced by the Hardy space of the polydisc $H^2(\mathbf{D}^n)$, or the Bergman space $L_a^2(\Omega)$ of an open set $\Omega \subset \mathbf{C}^n$. Another example is the ring $\mathcal{O}(\bar{\mathbf{B}})$ of analytic functions in a neighborhood of the unit ball $\mathbf{B} \subset \mathbf{C}^n$ with the topology induced by $L_a^2(\mathbf{B})$.

Let us remark that the ring is usually not complete in this topology. In what follows we study properties of ideals that are closed in the topology τ .

Lemma 1.1. The radical of a closed ideal is closed.

Proof: Let I be closed, and $r(I)$ be its radical. By Proposition 7.14 in [2] there exists an integer k such that $r(I)^k \subset I$. Let f_n be a sequence of elements in $r(I)$ converging to some f . We want to prove that $f \in r(I)$. Since the multiplication is continuous in each variable, for every $g \in r(I)^{k-1}$, $f_n g \rightarrow f g$, hence $f g \in I$, since I is closed. This shows that in particular $f g \in I$ for every $g \in r(I)^{k-1}$. Repeating the argument we get $f f_n g \rightarrow f^2 g$ for any $g \in r(I)^{k-2}$, hence $f^2 g \in I$ for $g \in r(I)^{k-2}$. Inductively we get $f^r g \in I$, for $g \in r(I)^{k-r}$ and $0 \leq r \leq k$, so $f^k \in I$ which shows that $f \in r(I)$.

Since \mathcal{R} is Noetherian, every ideal $I \subset \mathcal{R}$ has a (minimal) primary decomposition $I = Q_1 \cap Q_2 \cap \dots \cap Q_m$ where each Q_i is P_i -primary for some prime ideal P_i . The ideals P_i , $1 \leq i \leq m$ are called the prime ideals belonging to I (or associated to I).

Theorem 1.1. If an ideal is closed, then every prime ideal belonging to it is closed.

Proof: If I is closed and $f \in \mathcal{R}$, then the ideal $(I : f) = \{g \in \mathcal{R}, gf \in I\}$ is closed. Indeed, if $g_n \in (I : f)$ and $g_n \rightarrow g$ then $g_n f \rightarrow gf$, so $gf \in I$ which shows that $g \in (I : f)$.

From Lemma 1.1. we get that $r((I : f))$ is closed for every f . By Theorem 4.5. in [2] every prime ideal belonging to I is of this form hence it is closed.

This gives a positive answer to a question raised in [5].

Remark. Since the closure of an ideal is an ideal, a maximal ideal is either closed or dense.

Given an ideal $J \subset \mathcal{R}$, the J -adic topology on \mathcal{R} is the topology determined by the powers of J , so in this topology the closure of a set $A \subset \mathcal{R}$ is $\cap_n (A + J^n)$. For more details the reader can consult [2].

Let

$$\mathcal{C} := \{\mathcal{M} \subset \mathcal{R}, \mathcal{M} \text{ maximal ideal and the } \mathcal{M}\text{-adic topology is weaker than } \tau\}.$$

We see that \mathcal{C} consists of those maximal ideals \mathcal{M} for which \mathcal{M}^n is dense for every integer n . The following result is a slightly modified version of Theorem 2.7 in [5].

Theorem 1.2. If an ideal I has the property that for every prime P belonging to it there is $\mathcal{M} \in \mathcal{C}$ with $P \subset \mathcal{M}$, then I is closed.

Proof: Let P_1, P_2, \dots, P_m be the prime ideals belonging to I , $P_i \subset \mathcal{M}_i$, $\mathcal{M}_i \in \mathcal{C}$. If $J = \mathcal{M}_1 \mathcal{M}_2 \dots \mathcal{M}_m$ then the J -adic topology is weaker than τ , so it suffices to prove that I is closed in the J -adic topology. By Krull's Theorem ([2, Theorem 10.7]) and Proposition 4.7 in [2] the latter is true if and only if $J + P_i \neq \mathcal{R}$, $1 \leq i \leq m$. This is clearly satisfied since $J + P_i \subset \mathcal{M}_i$, and the theorem is proved.

Definition. The pair (\mathcal{R}, τ) is said to satisfy Hilbert's Nullstellensatz if every ideal $I \subset \mathcal{R}$ is either dense, or there exists $\mathcal{M} \in \mathcal{C}$ with $I \subset \mathcal{M}$.

Let us remark that if (\mathcal{R}, τ) satisfies Hilbert's Nullstellensatz then any closed ideal is contained in a maximal closed ideal, which motivates the terminology. By Krull's

Theorem, (\mathcal{R}, τ) satisfies Hilbert's Nullstellensatz for every J -adic topology τ . The ring $\mathbf{C}[z]$ with the topology induced by $H^2(\mathbf{D})$ or $L_a^2(\mathbf{D})$ also satisfies this property. In [7], a class of strongly pseudoconvex domains Ω for which $\mathcal{O}(\bar{\Omega})$ with the topology induced by $L_a^2(\Omega)$ satisfies Hilbert's Nullstellensatz has been exhibited.

Lemma 1.2. If I and J are two dense ideals in \mathcal{R} then $I \cdot J$ is dense.

Proof: Let $f_n \rightarrow 1$, $n \rightarrow \infty$, $f_n \in I$. If $g \in J$ then $f_n g \rightarrow g$ which shows that $I \cdot J$ is dense in J , hence in \mathcal{R} .

Theorem 1.3. If (\mathcal{R}, τ) satisfies Hilbert's Nullstellensatz then an ideal $I \subset \mathcal{R}$ is closed if and only if every prime belonging to I is closed. Moreover, the closure of an ideal in \mathcal{R} is equal to the intersection of its primary components that are contained in closed maximal ideals.

Proof: If I is closed then every prime belonging to I is closed by Theorem 1.1. Conversely, if a prime belonging to I is closed, then it is not dense, so it is included in an $\mathcal{M} \in \mathcal{C}$. The fact that I is closed now follows from Theorem 1.2.

For the second part, let \bar{I} be the closure of I in \mathcal{R} , and let $I = \cap_{i=1}^m Q_i$ be a (minimal) primary decomposition of I , such that Q_1, Q_2, \dots, Q_r are included in maximal ideals that are in \mathcal{C} , hence closed, and $Q_{r+1}, Q_{r+2}, \dots, Q_m$ are not, so they are dense. Then from the first part of the proof we get $\bar{I} \subset \cap_{i=1}^n Q_i$.

On the other hand, by Lemma 1.2 the ideal $Q_{r+1}Q_{r+2} \cdots Q_m$ is dense in \mathcal{R} , hence $Q_{r+1}Q_{r+2} \cdots Q_m(Q_1 \cap Q_2 \cap \cdots \cap Q_r)$ is dense in $Q_1 \cap Q_2 \cap \cdots \cap Q_r$, hence I is dense in $Q_1 \cap Q_2 \cap \cdots \cap Q_r$, so $\bar{I} = Q_1 \cap Q_2 \cap \cdots \cap Q_r$.

In [5] and [6] the authors state the following conjecture:

“Let $\mathcal{R} = \mathbf{C}[z_1, z_2, \dots, z_n]$ be endowed with the topology induced by $H^2(\mathbf{D}^n)$. Then an ideal I is closed if and only if every irreducible component of the zero set of I intersects \mathbf{D}^n .”

The results above show that the conjecture is equivalent to the topological Hilbert Nullstellensatz for the ring of polynomials with the topology induced by the Hardy space.

2. The case of the bidisk

In this section we prove the above mentioned conjecture for the case of two variables. Let us denote by \mathbf{T}^2 the 2-dimensional torus $\{(z_1, z_2) \in \mathbf{C}^2; |z_i| = 1\}$.

Lemma 2.1. If $\alpha \in \mathbf{C}$, $|\alpha| \geq 1$ and $1/2 < r < 1$ then for any z with $|z| = 1$ we have $|(z - \alpha)/(rz - \alpha)| \leq 2$.

Proof: The result follows from $|z - \alpha| \leq |z - \alpha/r|$. The last inequality is obvious since in the triangle formed by the points z, α and α/r the angle at α is obtuse.

Lemma 2.2. Let $p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$ be such that $|z_i| \geq 1$. If $1/2 < r < 1$ then for any z with $|z| = 1$, $|p(z)/p(rz)| \leq 2^n$.

Proof: The result follows by applying the previous lemma to each of the factors in the decomposition of p .

Proposition 2.1. Let $p \in \mathbf{C}[z_1, z_2]$ be a polynomial having no zeros in \mathbf{D}^2 . Then $pH^2(\mathbf{D}^2)$ is dense in $H^2(\mathbf{D}^2)$.

Proof: Since every polynomial is a product of irreducible polynomials, it suffices to prove the result for an irreducible polynomial. Let us show that 1 is in the closure of $pH^2(\mathbf{D}^2)$. Without loss of generality we can assume that $p \notin \mathbf{C}[z_2]$ (the case when p is constant being obvious). Write $p(z_1, z_2) = a_m z_1^m + \cdots + a_1 z_1 + a_0$ where $a_k \in \mathbf{C}[z_2]$. If we fix z_2^0 on the circle, the polynomial in z_1 , $p(z_1, z_2^0)$, does not vanish identically; for otherwise it would be divisible by $z_2 - z_2^0$, contradicting the irreducibility. So, for each z_2^0 on the unit circle there is a maximal $k \leq m$ such that $a_k(z_2^0) \neq 0$. It follows that the polynomial p has no zero (z_1^0, z_2^0) with $|z_1^0| < 1$ and $|z_2^0| = 1$ since otherwise, by slightly perturbing z_2 to a point in \mathbf{D} and using the continuous dependence of the roots on the coefficients, we could produce a zero for p in \mathbf{D}^2 . This shows that if $0 < r < 1$ the polynomial $p(rz_1, z_2)$ has no zeros in \mathbf{D}^2 , thus $1/p(rz_1, z_2) \in H^2(\mathbf{D}^2)$.

Consider the sequence $f_k(z_1, z_2) := p(z_1, z_2)/p((1 - 1/k)z_1, z_2) \in pH^2(\mathbf{D}^2)$. We want to show that $f_k \rightarrow 1$ in $H^2(\mathbf{D}^2)$. Fix z_2^0 on the unit circle and consider the maximal

$k \leq m$ such that $a_k(z_2^0) \neq 0$. By Lemma 2.2 we have $|f_k(z_1, z_2^0)| \leq 2^k$ for any z_1 on the unit circle, hence $|f_k(z_1, z_2)| \leq 2^m$ for $(z_1, z_2) \in \mathbf{T}^2$.

Let us denote by $V(p)$ the zero set of p . Since the 2-torus is not a complex algebraic variety, the set $V(p) \cap \mathbf{T}^2$ has real dimension strictly less than 2, thus it has measure zero. For $\epsilon > 0$, let us choose W a neighborhood of this set, whose Lebesgue measure on the torus is equal to $\epsilon/(2(2^m + 1)^2)$. Since $f_k \rightarrow 1$ uniformly on $\mathbf{T}^n \setminus W$ we can choose $N \in \mathbf{N}$ such that for $k > N$, $\|f_k - 1\|_{2, \mathbf{T}^n \setminus W}^2 < \epsilon/2$. It follows that for $k > N$, $\|f_k - 1\|_2^2 < (2^m + 1)^2 \epsilon / (2(2^m + 1)^2) + \epsilon/2 = \epsilon$, which proves the assertion.

Remark. We see that the only nontrivial situations when this result applies are those when the zero set of the polynomial touches the boundary of \mathbf{D}^n . Here are some examples of such polynomials: $z_1 z_2 - 1$, $z_1 + z_2 - 2$, $2z_1 z_2 + z_1 + z_2 + 2$.

Theorem 2.1. The ring $\mathbf{C}[z_1, z_2]$, with the topology induced by the Hardy space, satisfies the topological Hilbert Nullstellensatz.

Proof: Let us first prove the property for prime ideals. Using the classical Hilbert Nullstellensatz we see that the only maximal ideals in \mathcal{C} are those corresponding to points in \mathbf{D}^2 . Moreover, the other maximal ideals are dense. So by Theorem 1.2 we only have to show that if P is prime and $V(P) \cap \mathbf{D}^2 = \emptyset$, where $V(P)$ is the zero set of P , then P is dense in $\mathbf{C}[z_1, z_2]$. Standard results in dimension theory (see [2]) show that P is either maximal or principal. Indeed the maximal length of a chain of nonzero prime ideals containing P is 2. If we have $P_0 \subset P$ then P is maximal and the result follows easily. If $P \subset P_1$ then P must be principal since if P is generated by g_1, g_2, \dots, g_k we can take g_1 to be irreducible (using the fact that P is prime), and then the ideal generated by g_1 is included in P , so it must coincide with P . The density in the second case follows from Theorem 2.1.

If I is an arbitrary ideal having no zeros in \mathbf{D}^2 , let us show that it is dense. If P_1, P_2, \dots, P_n are the primes associated to it then from what has been established above it follows that they are dense. By Proposition 7.14 in [2] there exists an integer k such

that $(P_1 P_2 \cdots P_n)^k \subset I$. It follows from Lemma 1.2 that I itself is dense in $\mathbf{C}[z_1, z_2]$, which proves the theorem.

As a direct consequence of this result and Theorem 1.3 we get

Corollary 2.1. Let $\mathbf{C}[z_1, z_2]$ be endowed with the topology induced by the Hardy space. Then an ideal is closed if and only if each of the irreducible components of its zero set intersects \mathbf{D}^2 .

3. Ideals of finite codimension

Now let us suppose that \mathcal{R} is also a k -algebra, where k is an algebraically closed field, and that the scalar multiplication is continuous (not only separately continuous). By the classical Hilbert Nullstellensatz ([2, Corollary 5.24]), $\mathcal{R}/\mathcal{M} \cong k$ for every maximal ideal \mathcal{M} . Let us also assume that the family \mathcal{C} defined above consists of all closed maximal ideals. Thus in this case a maximal ideal \mathcal{M} is either dense, or the \mathcal{M} -adic topology is weaker than the topology of \mathcal{R} .

Example. If we consider $\mathbf{C}[z_1, z_2, \dots, z_n]$ with the topology induced by $H^2(\mathbf{D}^n)$ then the condition above is satisfied. In this case we have $\mathcal{C} = \mathbf{D}^n$.

In a similar way as we proved Proposition 1.2 we can establish the following result.

Lemma 3.1. Given an ideal I whose associated prime ideals are maximal, I is closed if and only if every maximal ideal belonging to I is closed.

Lemma 3.2. An ideal $I \subset \mathcal{R}$ has finite codimension in \mathcal{R} if and only if every prime ideal belonging to I is maximal.

Proof: Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$, Q_i \mathcal{M}_i -primary. By Proposition 7.14 in [2] there exists an integer n such that $\mathcal{M}_i^n \subset Q_i$, so $(\mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_m)^n \subset I$. Since $\dim \mathcal{R}/(\mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_m)^n < \infty$, I has finite codimension.

For the converse, let P be a prime ideal belonging to I . Then P has finite codimension as well, so \mathcal{R}/P is an integral domain that is finite over k , and since k is algebraically closed we must have $\mathcal{R}/P = k$, therefore P is maximal.

Let $\widetilde{\mathcal{R}}$ be the closure of \mathcal{R} in the topology τ . Since the multiplication is only separately continuous, $\widetilde{\mathcal{R}}$ is no longer a ring, but it is an \mathcal{R} -module. Each element $x \in \mathcal{R}$ induces a continuous multiplication morphism T_x on $\widetilde{\mathcal{R}}$. We shall denote by \widetilde{I} the closure in $\widetilde{\mathcal{R}}$ of an ideal I in \mathcal{R} , to avoid confusion with \bar{I} , the closure of I in \mathcal{R} . Clearly \widetilde{I} is a closed submodule of $\widetilde{\mathcal{R}}$. Also, if $Y \subset \widetilde{\mathcal{R}}$ is a closed submodule, then $Y \cap \mathcal{R}$ is an ideal that is closed in \mathcal{R} .

Definition.(see [2, page 58]) A submodule $Y \subset \widetilde{\mathcal{R}}$ is called primary in $\widetilde{\mathcal{R}}$ if $Y \neq \widetilde{\mathcal{R}}$ and every zero-divisor in $\widetilde{\mathcal{R}}/Y$ is nilpotent.(An element $x \in \mathcal{R}$ is called zero-divisor if the morphism induced by T_x on $\widetilde{\mathcal{R}}/Y$ has nontrivial kernel, and nilpotent if this morphism is nilpotent).

Remark. If $Y \subset \widetilde{\mathcal{R}}$ is primary then $(Y : \widetilde{\mathcal{R}}) = \{x \in \mathcal{R} \mid T_x \widetilde{\mathcal{R}} \subset Y\}$ is primary, so $P = r((Y : \widetilde{\mathcal{R}}))$ is prime. We say that Y is P -primary. Moreover, $(Y : \widetilde{\mathcal{R}}) = Y \cap \mathcal{R}$ so $Y \cap \mathcal{R}$ is also P -primary.

Although every ideal in \mathcal{R} has a primary decomposition, this is not true in general for the submodules of $\widetilde{\mathcal{R}}$. The next result shows that this is true for closed submodules of $\widetilde{\mathcal{R}}$ of finite codimension.

Theorem 3.1. There is a one-to-one correspondence between ideals in \mathcal{R} whose associated prime ideals are maximal and closed in \mathcal{R} , and closed submodules in $\widetilde{\mathcal{R}}$ of finite codimension, given by the maps $I \rightarrow \widetilde{I}$ and $Y \rightarrow Y \cap \mathcal{R}$. Moreover, if $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ is a (minimal) primary decomposition for I , then $\widetilde{I} = \widetilde{Q}_1 \cap \widetilde{Q}_2 \cap \cdots \cap \widetilde{Q}_m$ is a (minimal) primary decomposition for \widetilde{I} .

Proof: By Lemmas 3.1 and 3.2 we have to show that the maps indicated above establish a one-to-one correspondence between ideals of finite codimension that are closed in the topology of \mathcal{R} and closed submodules of finite codimension in $\widetilde{\mathcal{R}}$.

If Y is a closed submodule of $\widetilde{\mathcal{R}}$ of finite codimension then $\mathcal{R}/(Y \cap \mathcal{R}) \cong \widetilde{\mathcal{R}}/Y$ since the canonical map $\mathcal{R} \rightarrow \widetilde{\mathcal{R}}/Y$ is surjective, \mathcal{R} being dense in $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{R}}/Y$ being finite dimensional, and the kernel of this map is $Y \cap \mathcal{R}$. On the other hand $\widetilde{\mathcal{R}}/(Y \cap \mathcal{R}) \cong \mathcal{R}/(Y \cap \mathcal{R})$

\mathcal{R}), hence $Y = (\widetilde{Y \cap \mathcal{R}})$. Also for every ideal $I \subset \mathcal{R}$ that is closed in the topology of \mathcal{R} , $\widetilde{I \cap \mathcal{R}} = I$, so the two maps are inverses of one another, and the one-to-one correspondence is proved.

Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ be a primary decomposition of I . Then $\widetilde{I} \subset \widetilde{Q_1 \cap Q_2 \cap \cdots \cap Q_m}$, and since $\widetilde{I \cap \mathcal{R}} = \widetilde{Q_1 \cap Q_2 \cap \cdots \cap Q_m \cap \mathcal{R}} = Q_1 \cap Q_2 \cap \cdots \cap Q_m = I$, by the first part of the proof the two must be equal.

In the commutative diagram below the horizontal arrows are isomorphisms

$$\begin{array}{ccc} \mathcal{R}/Q_i & \xrightarrow{\sim} & \widetilde{\mathcal{R}}/\widetilde{Q_i} \\ T_x \downarrow & & \downarrow T_x \\ \mathcal{R}/Q_i & \xrightarrow{\sim} & \widetilde{\mathcal{R}}/\widetilde{Q_i} \end{array}$$

so the fact that Q_i is a primary ideal (hence a primary \mathcal{R} -module as well) implies that $\widetilde{Q_i}$ is a primary submodule of $\widetilde{\mathcal{R}}$.

If the primary decomposition of I is minimal let us show that the corresponding decomposition for \widetilde{I} is also minimal. Suppose that there exists j such that $\widetilde{I} = \widetilde{Q_1} \cap \cdots \cap \widetilde{Q_{j-1}} \cap \widetilde{Q_{j+1}} \cap \cdots \cap \widetilde{Q_m}$. Then $I = \widetilde{I \cap \mathcal{R}} = Q_1 \cap \cdots \cap Q_{j-1} \cap Q_{j+1} \cap \cdots \cap Q_m$, which contradicts the minimality of the primary decomposition of I , and the theorem is proved.

From the proof it follows that the associated primes of I and \widetilde{I} coincide. Corollary 4.11 in [2] shows that in this case the minimal primary decomposition is unique.

Remark. In the case when the topology on \mathcal{R} comes from a norm, the first part of the theorem is contained in [5, Corollary 2.8].

4. Examples

In this last section we shall describe some examples to which Theorem 3.1 can be applied.

1. For the case $\mathcal{R} = \mathbf{C}[z_1, z_2, \dots, z_n]$ and $\widetilde{\mathcal{R}} = H^2(\mathbf{D}^n)$ the theorem has been proved by Ahern and Clark in [1]. The primary closed $\mathbf{C}[z_1, z_2, \dots, z_n]$ -submodules of finite

codimension of $H^2(\mathbf{D}^n)$ are those closed submodules Y for which there exists a point $\lambda \in \mathbf{D}^n$ and a number $m \in \mathbf{N}$ such that Y contains the space of functions f that satisfy $(\partial^m / \partial z_1^m \partial^m / \partial z_2^m \cdots \partial^m / \partial z_n^m f)(\lambda) = 0$.

2. If \mathbf{B} is the unit ball in \mathbf{C}^n , $\mathcal{R} = \mathcal{O}(\bar{\mathbf{B}})$ and $\widetilde{\mathcal{R}} = L_a^2(\mathbf{B})$ then $\mathcal{C} = \mathbf{B}$, and if \mathcal{M} is a maximal ideal corresponding to a point in $\partial\mathbf{B}$ then \mathcal{M} is dense in $L_a^2(\mathbf{B})$ (see [7]). This shows that the hypothesis of Theorem 2.1 is satisfied. The primary closed $\mathcal{O}(\bar{\mathbf{B}})$ -modules of finite codimension have the same description as in the previous example.

3. The last example concerns a certain class of Reinhardt domains. Let $0 < p, q < \infty$. Following [4] we define

$$\Omega_{p,q} = \{z \in \mathbf{C}^2 \mid |z_1|^p + |z_2|^q < 1\}.$$

We shall prove that $\mathbf{C}[z_1, z_2]$ with the topology induced by the L^2 -norm on $\Omega_{p,q}$ also satisfies the hypothesis of Theorem 3.1.

Lemma 4.1. If $0 \leq a \leq 1$ then the series

$$\sum_{m,n \geq 0} \binom{m+n}{m} a^m (1-a)^n$$

diverges.

Proof: By symmetry, we can assume $a < 1$. We have

$$\begin{aligned} \sum_{m,n \geq 0} \binom{m+n}{m} a^m (1-a)^n &= \sum_{p \geq 0} \sum_{q \leq p} \binom{p}{q} a^q (1-a)^{p-q} = \\ &= \sum_{p \geq 0} (1-a)^p \sum_{q \leq p} \binom{p}{q} \left(\frac{a}{1-a}\right)^q = \sum_{p \geq 0} (1-a)^p \cdot \frac{1}{(1-a)^p} = \infty \end{aligned}$$

Corollary 4.1. If $0 \leq a \leq 1$ then the series

$$\sum_{m,n > 0} \binom{2(m+n)-2}{2m-1} a^{2m-1} (1-a)^{2n-1}$$

diverges.

Let B denote the beta function, defined by $B(r, s) := \Gamma(r)\Gamma(s)/\Gamma(r+s)$, $r, s > 0$.
By [3],

$$B(r, s) = \int_0^1 t^{r-1}(1-t)^{s-1} dt.$$

Lemma 4.2. If $(z_1, z_2) \in \partial\Omega_{p,q}$ then the series

$$\sum_{r_1, r_2 \geq 0} \frac{|z_1|^{2r_1} |z_2|^{2r_2}}{B(\frac{2r_1+2}{p}, \frac{2r_2+2}{q} + 2)}$$

diverges.

Proof: Since $B(r, s)$ is a decreasing function in r and s , by taking the subseries corresponding to indices with the property that $pm \leq r_1 \leq pm+1$ and $qn \leq r_2 \leq qn+1$ we get the inequality

$$\sum_{r_1, r_2 \geq 0} \frac{|z_1|^{2r_1} |z_2|^{2r_2}}{B(\frac{2r_1+2}{p}, \frac{2r_2+2}{q} + 2)} \geq |z_1|^{p+2} |z_2|^{q+2} \sum_{m, n \geq 0} \frac{|z_1|^{(2m-1)p} |z_2|^{(2n-1)q}}{B(2m, 2n)}.$$

Since $|z_1|^q = 1 - |z_2|^p$ the latter series is equal to $\sum_{m, n \geq 0} (|z_1|^p)^{2m-1} (1 - |z_1|^p)^{2n-1} / B(2m, 2n)$.
Applying Corollary 4.1 for $a = |z_1|^p$, and using the fact that $B(2m, 2n) = (2m-1)!(2n-1)!/(2m+2n-1)! < 1/\binom{2(m+n)-2}{2m-1}$ (see [3]) we get the desired result.

Proposition 4.1. If we endow the ring $\mathbf{C}[z_1, z_2]$ with the topology induced by $L^2(\Omega_{p,q})$, then a maximal ideal $\mathcal{M} \subset \mathbf{C}[z_1, z_2]$ is either dense, or the \mathcal{M} -adic topology is weaker than the topology of the ring.

Proof: By [4, Theorem 4.9 c) and Example 5.2], an ideal $\mathcal{M} = (z_1 - w_1, z_2 - w_2)$ is dense in $\mathbf{C}[z_1, z_2]$ with the L^2 -norm if and only if

$$\frac{2(\pi)^2}{p} \sum_{r_1, r_2 \geq 0} (r_2 + 1) \frac{|w_1|^{2r_1} |w_2|^{2r_2}}{B(\frac{2r_1+2}{p}, \frac{2r_2+2}{q} + 1)}.$$

diverges.

By Lemma 4.2 this series diverges if $(w_1, w_2) \in \partial\Omega_{p,q}$, so if $w_1, w_2 \notin \Omega_{p,q}$ the ideal \mathcal{M} is dense.

It is not difficult to check that \mathcal{M}^m is closed whenever $(w_1, w_2) \in \Omega_{p,q}$ and $m \in \mathbf{N}$. Therefore $\mathcal{C} = \Omega_{p,q}$, and if $\mathcal{M} \notin \mathcal{C}$ then \mathcal{M} is dense.

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Department of Mathematics, The University of Iowa, Iowa City, IA 52242

E-mail: rgelca@math.uiowa.edu

and

Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, Bucharest 70700, Romania.